Lipschitz Shadowing and Structural Stability

Sergey Tikhomirov\textsuperscript{1}  Sergei Pilyugin\textsuperscript{2}  Alexey Osipov\textsuperscript{2}

\textsuperscript{1}National Taiwan University

\textsuperscript{2}Saint-Petersburg State University

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Standard Shadowing and Structural Stability

- $f : M \rightarrow M$, $f \in C^1$, $M \in C^\infty$, dist.
- $\{\xi_n\}$ is $d$-pseudotrajectory, if $\text{dist}(\xi_{n+1}, f(\xi_n)) < d$

- Standard Shadowing (StSh)
  $\forall \varepsilon > 0 \exists d > 0$ such that $\forall$ $d$-pseudotrajectory $\{\xi_n\}$ there exists exact trajectory $\{x_n\}$ such that
  \[\text{dist}(x_n, \xi_n) < \varepsilon.\]

- SS – set of structurally stable diffeomorphisms.
  There exists neighborhood $U_f$ in the $C^1$-topology such that for any $g \in U_f$, diffeomorphisms $f$ and $g$ are topologically conjugated.
Known Facts and Hypothesis

- \( SS \subset StSh \) (Robinson 1977, Sawada 1980).
  - Shadowing lemma: If \( \Lambda \) is hyperbolic then \( f \) has shadowing in some \( U(\Lambda) \).
- \( SS \not= StSh \).
- \( Int^1(StSh) = SS \) (Sakai, 1994).
- Hypothesis Abdenur-Diaz: generically \( StSh = SS \).
- Shadowing \( \Rightarrow \) ? Structural Stability.
**Lipschitz Shadowing**

- **Lipschitz Shadowing (LipSh)**
  \[ \exists L, d_0 > 0 \text{ such that } \forall d < d_0 \text{ and } d\text{-pseudotrajectory } \{\xi_n\} \]
  there exists exact trajectory \( \{x_n\} \) such that
  \[ \text{dist}(x_n, \xi_n) < Ld. \]

**Theorem (Pilyugin, Tikhomirov, 2009)**

\( SS = \text{LipSh}. \)

- **Expansivity (EXP):** \( \exists a > 0 \text{ such that if } \forall n \in \mathbb{Z} \]
  \[ \text{dist}(f^n(x), f^n(y)) < a \text{ then } x = y. \]
- \( SS \cap \text{EXP} = \text{Anosov} \) (Mane, 1974).
  - Anosov diffeomorphism – whole manifold is a hyperbolic set.

**Consequence**

\( \text{LipSh} \cap \text{EXP} = \text{Anosov}. \)
Periodic Shadowing

- $\Omega S$ – set of $\Omega$-stable diffeomorphisms
  There exists neighborhood $U_f$ in the $C^1$-topology such that for any $g \in U_f$, $f$ is topologically conjugated to $g$ on $\Omega(f)$.

- Periodic Shadowing (PerSh)
  $\forall \varepsilon > 0 \ \exists d > 0$ such that $\forall$ periodic $d$-pseudotrajectory $\{\xi_n\}$ there exists periodic exact trajectory $\{x_n\}$ such that
  $$\text{dist}(x_n, \xi_n) < \varepsilon.$$  

- Lipschitz Periodic Shadowing (LipPerSh)
  $\varepsilon = Ld$.  

Theorems (Osipov, Pilyugin, Tikhomirov, 2009)

- LipPerSh $= \Omega S$.
- $\text{Int}^1(\text{PerSh}) = \Omega S$.  

Proof of LipSh $= SS$

\textbf{Mane, 1977:}

$$E^s(x) = \{ v \in T_xM, \ Df^n(x)v \to_{n \to +\infty} 0 \}$$

$$E^u(x) = \{ v \in T_xM, \ Df^n(x)v \to_{n \to -\infty} 0 \}$$

\textbf{Theorem:} If $\forall x \in M \ E^s(x) \oplus E^u(x) = T_xM$ then $f \in SS$.

\textbf{Pliss, 1977:}

$p_n$ – exact trajectory, $A_n = Df(p_n)$.

$$v_{n+1} = A_n v_n + w_n.$$ 

\textbf{Theorem:} If $\forall |w_n| < 1$ there exists $|v_n| < \infty$ then

$$E^s(p_n) \oplus E^u(p_n) = T_{p_n}M.$$
Proof of $\text{LipPerSh} = \Omega S$

- $f \in \text{LipPerSh}$, Lipschitz constant $L > 0$.
- Periodic orbits are hyperbolic.
- Periodic orbits are uniformly hyperbolic
  - $p_n$ – periodic trajectory, $\nu \in E^u(p_0)$.
  - 
    \[ |D f^k(p_0)\nu| \geq \frac{1}{L} \left(1 + \frac{1}{L}\right)^{k-1}, \quad k > 0.\]
  - Periodic orbits are dense in $\Omega(f)$.
  - Passing to a limit we prove Axiom A.
  - “no-cycle” condition.
## Conclusion

### Theorems
- LipSh = SS.
- LipPerSh = ΩS.

### Conclusion
Lipschitz Shadowing $\Rightarrow$ Hyperbolicity.

### Main Idea
\[ f(x) \leftrightarrow v_{n+1} = A_n v_n. \]
Thank you very much for your attention!