Lipschitz shadowing for flows

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Compare shadowing and structural stability

- Case of diffeomorphisms.
- Definitions for flows.
- Future questions, partially hyperbolic dynamics.
- Proof for flows.
Shadowing for diffeomorphisms

- \( f : M \to M, f \in C^1, M \in C^\infty, \text{ dist.} \)
- \( \{ \xi_n \} \text{ is } d\text{-pseudotrajectory, if } \text{dist}(\xi_{n+1}, f(\xi_n)) < d \)

Standard Shadowing (StSh)
\( \forall \varepsilon > 0 \, \exists d > 0 \text{ such that } \forall \text{ } d\text{-pseudotrajectory } \{ \xi_n \} \text{ there exists exact trajectory } \{ x_n \} \text{ such that } \)
\[ \text{dist}(x_n, \xi_n) < \varepsilon. \]

Lipschitz Shadowing (LipSh)
\( \exists L, d_0 > 0 \text{ such that } \forall d < d_0 \text{ and } d\text{-pseudotrajectory } \{ \xi_n \} \text{ there exists exact trajectory } \{ x_n \} \text{ such that } \)
\[ \text{dist}(x_n, \xi_n) < Ld. \]
Lipschitz Shadowing and structural stability

- \( SS \) – set of structurally stable diffeomorphisms.
  \( SS = \text{Axiom A} + \text{Strong Transversality Condition} \).
- Shadowing lemma: If \( \Lambda \) is hyperbolic then \( f \) has Lipschitz shadowing and expansivity in some \( U(\Lambda) \).
- \( \text{Int}^1(\text{StSh}) = SS \) (Sakai, 1994).
- Conjecture Abdenur-Diaz: generically \( \text{StSh} = SS \) (proved for tame diffeomorphisms).

Robinson 1977, Sawada 1980; Pilyugin, Tikhomirov 2010

\( \text{LipSh} = SS \).

- Expansivity (EXP): \( \exists \ a > 0 \) such that if \( \forall n \in \mathbb{Z} \)
  \( \text{dist}(f^n(x), f^n(y)) < a \) then \( x = y \).
- Consequence: \( \text{LipSh} \cap \text{EXP} = Anosov \).
Periodic Shadowing

- Periodic Shadowing (PerSh)
  \[ \forall \varepsilon > 0 \ \exists \ d > 0 \text{ such that } \forall \text{ periodic } d\text{-pseudotrajectory } \{\xi_n\} \]
  there exists periodic exact trajectory \{x_n\} such that
  \[ \text{dist}(x_n, \xi_n) < \varepsilon. \]

- Lipschitz Periodic Shadowing (LipPerSh)
  \[ \varepsilon = Ld. \]

- \( \Omega S \) – set of \( \Omega \)-stable diffeomorphisms
  Axiom A + no-cycle condition.

Theorems (Osipov, Pilyugin, Tikhomirov, 2010)

- \( \text{LipPerSh} = \Omega S. \)
- \( \text{Int}^1(\text{PerSh}) = \Omega S. \)
Main step of the proof LipSh $\Rightarrow$ SS.

$p_n$ – exact trajectory, $A_n = D f(p_n)$.

**Exponential dichotomy**

Sequence $A_n$ has exponential dichotomy on $\mathbb{R}^+$ if there exists decomposition $T_{p_n} = S_n^+ \oplus U_n^+$ for $n \geq 0$ and $C > 0$, $\lambda \in (0, 1)$ such that

1. $A_n S_n^+ = S_{n+1}^+$, $A_n U_n^+ = U_{n+1}^+,$
2. $|A_{n+k-1} \cdots A_n v_s| \leq C \lambda^k |v_s|$, for $v_s \in S_n^+$
3. $|A_{n+k-1} \cdots A_n v_u| \geq \frac{1}{C} \lambda^{-k} |v_u|$, for $v_u \in U_n^+$

For $\mathbb{R}^-$ denote splitting $T_{p_n} = S_n^- \oplus U_n^-$
Main steps of the proof LipSh $\Rightarrow$ SS.

Pliss, 1977:
If $\forall |b_n| < 1$ there exists $|v_n| < L < \infty$, satisfying

$$v_{n+1} = A_n v_n + b_n.$$ 

then $A_n$ has exponential dichotomy on $\mathbb{R}^+$ and $\mathbb{R}^-$ and $S_0^+ + U_0^- = T_{p_0}$. 
Shadowing for Vector fields

- Vector field $X$ on $M$. Flow $\varphi(t, x) : \mathbb{R} \times M \rightarrow M$.
- Map $g(t) : \mathbb{R} \rightarrow M$ is $d$-pseudotrajectory, if
  \[ \text{dist}(g(t + \tau), \varphi(\tau, g(t))) < d, \quad \text{for} \quad t \in \mathbb{R}, |\tau| < 1 \]

- **Standard Shadowing property** ($StSh$)
  \[ \forall \varepsilon \exists d \text{ such that } \forall d\text{-pseudotrajectory } g(t) \text{ there exists } x_0 \in M \text{ and reparametrization } h(t) : \mathbb{R} \rightarrow \mathbb{R}, \text{satisfying} \]
  \[ \text{dist}(g(t), \varphi(h(t), x_0)) < \varepsilon, \quad t \in \mathbb{R} \]
  \[ \left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1 \right| < \varepsilon, \quad t_1, t_2 \in \mathbb{R}, t_1 \neq t_2 \]
Why we need reparametrization

\[ f(x) = \varphi(1, x) \]
No reparametrization ⇔ \( f \) has shadowing property
Lipschitz Shadowing for flows

**Lipschitz Shadowing property (LipSh)**
\[ \exists L, d_0 \ \forall d < d_0 \text{ and } d \text{-pseudotrajectory } g(t) \text{ exists } x_0 \in M \text{ and } \]
reparametrization \( h(t) : \mathbb{R} \to \mathbb{R} \), satisfying the inequalities
\[
\left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1 \right| < Ld, \quad t_1, t_2 \in \mathbb{R}, t_1 \neq t_2,
\]
\[
\text{dist}(g(t), \varphi(h(t), x_0)) < Ld, \quad t \in \mathbb{R}.
\]

**Lipschitz Periodic Shadowing property (LipPerSh)**
\[ \exists L, d_0 \ \forall d < d_0 \text{ and periodic } d \text{-pseudotrajectories } g(t) \text{ exist } x_0 \in M \]
and reparametrization \( h(t) : \mathbb{R} \to \mathbb{R} \), satisfying the inequalities
\[
\left| \frac{h(t_1) - h(t_2)}{t_1 - t_2} - 1 \right| < Ld, \quad t_1, t_2 \in \mathbb{R}, t_1 \neq t_2,
\]
\[
\text{dist}(g(t), \varphi(h(t), x_0)) < Ld, \quad t \in \mathbb{R}.
\]
and \( x_0 = \phi(\tau, x_0) \) for some \( \tau > 0 \).
Main Results

- $\Omega S$ – set of omega stable vector fields. Axiom A’ + No-cycle condition.
- $SS \subset \text{LipSh}$ (Pilyugin, 1997)
- $\text{Int}^1(StSh + \text{“no singularities”}) \subset SS$ (Lee, Sakai, 2008)
- $\text{Int}^1(StSh)? =? SS, \quad \text{Int}^1(OrientSh) \neq SS$ (Pilyugin, Tikhomirov, 2010)

Main Results (Palmer, Pilyugin, Tikhomirov)

\[
\text{LipSh} = SS. \\
\text{LipPerSh} = \Omega S.
\]
Future questions

- Let $f$ be a partially hyperbolic, dynamically coherent diffeomorphism, $W^c$ - central foliation.

- $d$-pseudotrajectory – a sequence $\{y_k\}$, satisfying
  \[
  \text{dist}(y_{k+1}, f(y_k)) < d.
  \]

- $d$-central pseudotrajectory – a sequence $\{y_k\}$ which is $d$-pseudotrajectory and
  \[
  y_{k+1} \in W^c_{loc}(f(y_k)).
  \]

- **Conditional shadowing:** $\forall \varepsilon > 0 \ \exists d > 0$ such that for any $d$-pseudotrajectory $\{y_k\}$ there exists $\varepsilon$-central pseudotrajectory $\{x_k\}$, such that
  \[
  \text{dist}(x_k, y_k) < \varepsilon.
  \]

- **Plaque expansivity:** $\exists a > 0$ such that if $\{x_k\}, \{y_k\}$ are $a$-central pseudotrajectory and $\text{dist}(x_k, y_k) < a$, then $x_k \in W^c_{loc}(y_k)$
Future questions

- Can one repeat shadowing theory for partially hyperbolic case?
- Perfectly matches with the notion of plaque expansivity.
- Holds for shift-one map.
- Is it important?
Scheme of the proof LipSh $\Rightarrow$ SS.

Steps:
1. Fixed points are hyperbolic.
2. Fixed points are isolated in the chain-recurrent set.
3. Non-singular part of the chain-recurrent set is hyperbolic (Axiom A’)
4. The strong transversality condition.

Tool:
Inhomogeneous linear equation
Discrete Lipschitz shadowing property

Let \( f(x) = \phi(1, x) \).

Discrete Lipschitz shadowing property

There exist \( d_0, L > 0 \) such that if \( y_k \in M \) is a sequence with

\[
\text{dist}(y_{k+1}, f(y_k)) \leq d, \quad k \in \mathbb{Z}
\]

for \( d \leq d_0 \), then there exist sequences \( x_k \in M \) and \( t_k \in \mathbb{R} \) satisfying

\[
\text{dist}(x_k, y_k) \leq Ld,
\]

\[
x_{k+1} = \phi(t_k, x_k),
\]

\[
|t_k - 1| \leq Ld.
\]

Remark: \( t_k \) – encapsulate non-hyperbolicity of \( f(x) \).
Main Lemma

- $x(t)$ – arbitrary trajectory of $X$.
- $p_k = x(k)$, $A_k = Df(p_k)$.
- Let $b_k \in T_{p_k}M$ be a bounded sequence, $b = \|b\|_\infty$.
- Consider equations

$$v_{k+1} = A_k v_k + b_{k+1}$$

- There exist sequences

$$s_k \in \mathbb{R}, \quad \text{with} \quad |s_k| \leq b' = L(2b + 1),$$

$$v_k \in T_{p_k}M \quad \text{with} \quad \|v\|_\infty \leq 2b'.$$

such that

$$v_{k+1} = A_k v_k + X(p_{k+1})s_k + b_{k+1}$$
Orthogonal projection

- Let $V_k \subset T_{p_k}$, such that $V_k \perp X(p_k)$.
- $P_k$ – orthogonal projection on $V_k$.
- $B_k = P_{k+1}A_k : V_k \mapsto V_{k+1}$.
- Let $b_k \in V_k$ be a bounded sequence, $b = \|b\|_\infty$.
- Then there exists a sequence

  $$v_k \in V_k, \quad \text{with} \quad \|v_k\| \leq 2L(2b + 1)$$

such that

  $$v_{k+1} = B_k v_k + b_{k+1}, \quad k \in \mathbb{Z}.$$
Scheme of the proof LipPerSh \(\Rightarrow\) SS.

**Steps:**
1. Fixed points are hyperbolic.
2. Fixed points are isolated in the chain-recurrent set.
3. Uniform hyperbolicity of closed trajectories
4. No-cycle condition.

**Tool:**
Inhomogeneous linear equation for periodic sequences
Thank you very much for your attention!